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13. ABSTRACT

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ABSTRACT

A theory of nonlinear global magnetohydrodynamic stability is described. The formalism is an entirely new approach to the problem. The concepts of space-time and generalized gauge symmetries of the flow fields are invoked to find constants of the motion. The constants correspond to charge operators in a theory of the current algebra of the fields. The charges, in turn, are defined by integrals that are determined by the symmetries of the fields. The strengths of the individual components of the currents determine the amount of symmetry breaking in each physical situation. The constants of the motion corresponding to the charge operator are used in conjunction with the principle of least constraint to generate the Euler-Lagrange equations corresponding to stable plasma motion. For every symmetry there is a corresponding conserved integral or charge (Noether's Theorem). The principle of least constraint states that if the total energy of the flow field of a bounded plasma cell is varied, subject

to a set of constraint integrals, then the fewer the number of constraint integrals applied, the more stable the resulting flow. The constraint integrals which generate a linear (superposable) field yield a set of equations which describe force-free collinear flow. If linearity is sacrificed, then fewer constraints can be used and many other types of flow structures are possible.

The symmetries and corresponding flow structures are classified by the Lie algebra of the currents and charges. The formalism leads to uniqueness theorems necessary to calculate the type of structure present for a given set of boundary conditions. It is demonstrated that a multiple integral variation problem can be related to the principle of least constraint. The fundamental variational formula for the appropriate tensor fields is developed. The group generators for a particular space-time and gauge transformation are then used to demonstrate that the variational approach of Woltjer and Wentzel is a special case of the problem of Lagrange with expanded integral constraints.

Proof that the theory actually describes the lowest lying and most stable energy states of the flow structures is given in a succeeding paper in the form of experimental data which is a measure of the actual magnetic fields trapped in the linear structures. Agreement with theory is excellent.

Section I - Introduction

In two previous papers (Wells and Norwood, 1969, and Wells, 1970) hereafter designated I and II respectively, a new method of calculating the structure of naturally occurring stable plasma cells was outlined. The method invokes a variational principle in which the total energy of a closed (bounded) plasma configuration (cell) is varied subject to a set of constraint integrals on the flow. The resulting Euler-Lagrange equations describe the magnetic and flow fields in the plasma "bunch" or plasmoid. These differential equations can be solved subject to appropriate boundary conditions to give a quantitative description of the structure of globally stable plasmoids; i.e., the stability is calculated for the whole plasmoid (plasma cell) in its actual geometric configuration and ambient plasma surroundings.

The low-lying cell energy states are analogous to the low-lying stable states of an atom in which the lowest lying and most stable atomic configurations correspond to minimum energy and the higher energy states correspond to structures that are more easily perturbed and broken down when subjected to external disturbances. The type of stable cell-state changes as one changes the type and number of constraint integrals used in the variational calculation.

The total energy is used in making the calculations because nearly all plasmoids (cells) are interacting and exchanging energy with their surroundings. This means that the structures are non-conservative systems, and the usual methods of stability calculation utilizing effective potentials have no meaning. The method is interesting because the calculations include a consideration of all nonlinear states, they are global and not local, they make no assumption about the strength of coupling between various modes in the plasma and include

the dynamic nonlinear terms in the equations of motion or equivalently in the equations describing conservation of formal currents and charges in the plasma.

The principle of least constraint (discussed in II) is invoked in order to find the various lowest lying cell energy states. The fewer the number of constraints applied to the system in performing the variational calculation, the more stable the corresponding plasmoid. It was also shown in II that the lowest lying states correspond, under certain conditions, to superposable flows and fields. All other states are non-superposable and nonlinear. Two of these nonlinear states will interact to form new states, rather than simply superpose to form a composite structure.

We will rederive the entire theory utilizing an approach that is more rigorous and physically more transparent than the methods utilized in I and II. The methods and formalisms of field theory will be used to derive an operator formalism that will enable us to extend the concept of the stable plasma states to a method of calculating the growth of motions and currents from an arbitrary initial plasma state.

There has been much discussion about the validity of this approach for calculating the stability of plasma confinement schemes. A second paper follows this one in which some of the predictions of the theory are verified in detail in the laboratory.

In this second paper (Nolting, Jindra and Wells, 1972), we will describe a plasma confinement scheme (TRISOPS) which utilizes all of these results to guide the production of high temperature, high density plasma with a long particle confinement time. Experiments will be described that compare the structure of the lowest lying plasma cell states with the theoretically predicted structure.

Section II outlines a derivation of the formal charges and currents, discusses the principle of least constraint and illustrates why the lowest lying linear cell states have minimum "free energy" available to drive "instabilities". Section III develops an operator formalism that can be used in a general attack on the problem of the growth and dynamics of interacting plasmoids (cells). Section IV applies the concept of broken symmetries to plasmoid interactions.

Section II - Formal Charges and Currents

In I and II the lowest lying stable plasma cell states are derived. The constants of the motion corresponding to space-time and gauge symmetries are derived using the Clebsch potentials and the concept of a "generalized gauge transformation". In this section we consider all of the constants of the motion corresponding to space-time and gauge symmetries. We discuss the expansions of the corresponding functionals and utilize them in a fundamental variational formula to show that minimum constraint corresponds to minimum total energy. We then show that minimum total energy corresponds to minimum "free energy" available to drive instabilities in the case of the lowest lying linear states.

In II we discussed the fundamental importance of Noether's theorem in relating the symmetry properties of the Lagrange density of the MHD fields to the constants of the motion for closed plasma structures. It is necessary to give a precise formulation of the basic problem of the calculus of variations if further progress is to be made in understanding the global stability problem for bounded plasma cells. We merely outline the necessary theorems. A rigorous treatment of the problem is given by Rund (Rund, 1966).

Given n real variables x^i, together with m independent real variables t^{α}, (Latin and Greek indices run from 1 to n and 1 to m respectively) consider the space R_{n+m} of n+m dimensions of x^i, t^{α}. Denote a subspace C_m in R_{n+m}.

A set of n equations of the type

$$x^i = x^i(t^\alpha) \tag{1}$$

defines this subspace. It is assumed that we can form the derivatives $\frac{\partial x^i}{\partial t^\alpha}$ on C_m. We denote these derivatives as

$$\dot{x}^i_\alpha (t^\beta) = \frac{\partial x^i(t^\beta)}{\partial t^\alpha} \quad .$$

Assume that G_t denotes a fixed simply-connected domain in the m-dimensional space of the t^α, bounded by a hypersurface ∂G_t. Each point of this surface corresponds to a set of values of the t^α. Consider a second set of equations of the type $\bar{x}^i = \bar{x}^i(t^\alpha)$ representing another subspace $\bar{C}_m$ of R_{n+m}. We require that this second subspace coincide with C_m for those values of t^α which define the boundary ∂G_t of G_t. Then

$$\bar{x}^i (t^\alpha) = x^i(t^\alpha) = f^i(t^\alpha) \text{ for } t^\alpha \in \partial G_t,$$

where the functions $f^i(t^\alpha)$ are separately prescribed.

Consider a suitably differentiable function $L=L(t^\alpha,x^j,\dot{x}^j_\alpha)$ with m+n+mn arguments. This function is defined as a function of t^α over each subspace of the type of Eq. (1).

We now form the integral

$$I(C_m) = \int_{G_t} L(t^\alpha,x^j,\dot{x}^j_\alpha)d(t), \tag{2}$$

where

$$d(t) \overset{\text{def}}{=} dt^1 \; ..... \; dt^m.$$

The value of Eq. (2) depends on C_m, i.e., the choice of the functions defined by Eq. (1) together with their derivatives. The fundamental problem is to find the necessary and sufficient conditions the $x^i(t^\alpha)$ must satisfy in order to yield an extreme value of the integral (2).

In order to proceed with the proof another set of n equations of the type

$$x^i = x^i(t^\alpha, u)$$

is considered. These represent a 1-parameter family of m-dimensional subspaces $C_m(n)$ of R_{n+m}. Consider two neighboring subspaces $C_m(u_0)$ and $C_m(u)$, where $(u-u_0)$ is considered to be small, and quantities of order $|u-u_0|^2$ are neglected. Let P and P' be points on $C_m(u_0)$ and $C_m(u)$ respectively, corresponding to the same t^α-values. The components of the displacement PP' in R_{n+m} are given by $(0,\ldots.0;\ \delta x^1,\ldots.,\ \delta x^m)$, where

$$\delta^* x^i = (u-u_0)\left(\frac{\partial x^i}{\partial u}\right)_{u=u_0} + \ldots \quad . \tag{3}$$

It can then easily be shown (Rund, 1966, p. 213) that

$$\delta x^i = (\dot{x}^i)_{u=u_0}\,\delta t^\alpha + \delta^* x^i \tag{4}$$

and

$$\delta^* \dot{x}^i = \frac{\partial}{\partial t^\alpha}(\delta^* x^i).$$

If one defines the first variation of I (as given by (2)) as

$$\delta I = I(u) - I(u_0) = (u-u_0)\left(\frac{dI}{du}\right)_{u=u_0} ,$$

then it can be shown that

$$\delta I = \int_{G_t} \left(\frac{\partial L}{\partial x^i}\,\delta^* x^i + \frac{\partial L}{\partial \dot{x}^i_\alpha}\,\delta^* \dot{x}^i_\alpha\right) d(t) + \int_{G_t} \frac{d}{dt^\alpha}\,(L\,\delta t^\alpha)\,d(t) .$$

This can be written as

$$\delta I = \int_{G_t}\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^i_\alpha}\right)\right]\delta^* x^i d(t) + \int_{G_t}\frac{d}{dt^\alpha}\left[L\,\delta t^\alpha + \frac{\partial L}{\partial \dot{x}^i_\alpha}\delta^* x^i\right]d(t), \qquad (5)$$

using Eq. (4).

The first variation takes the final form,

$$\delta I = \int_{G_t}\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^i_\alpha}\right)\right]\delta^* x^i d(t) + \int_{G_t}\frac{d}{dt^\alpha}\left\{\left(L\,\delta^\alpha_\beta - \frac{\partial L}{\partial \dot{x}^i_\alpha}\dot{x}^i_\beta\right)\delta t^\beta + \frac{\partial L}{\partial \dot{x}^i_\alpha}\delta x^i\right\}d(t) . \qquad (6)$$

Equations (5) and (6) are the fundamental variational formulas for multiple integrals. The integrand of the second integral is formally a divergence. It is now easily shown from Eq. (6) that an extremum for the integral defined by Eq. (2) is uniquely determined by the solutions of

$$\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^k_\alpha}\right) - \frac{\partial L}{\partial x^k} = 0 \quad ,$$

where

$$k = 1, \ldots\ldots, n.$$

This set of n equations reduces to the Euler-Lagrange equations for single integrals when m=1. They are the Euler-Lagrange equations for multiple integrals. Written out in full they take the form

$$\frac{\partial^2 L}{\partial \dot{x}^k_\alpha \partial \dot{x}^i_\beta}\frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} + \frac{\partial^2 L}{\partial x^i \partial \dot{x}^k_\alpha}\frac{\partial x^i}{\partial t^\alpha} + \frac{\partial^2 L}{\partial t^\alpha \partial \dot{x}^k_\alpha} - \frac{\partial L}{\partial x^k} = 0 \quad . \qquad (7)$$

It can be shown (Rund, 1966) how certain invariance properties of multiple integrals imply that various quantities are constant along an extremal. These results can be formulated in terms of Noether's Theorem. The so-called conservation laws then easily follow. Application of Noether's theorem to MHD global stability is discussed by Wells in II (Wells, 1970). Here we utilize the formalism of the fundamental symmetry transformations to directly tie together Noether's theorem and the principle of least constraint.

Consider an r-parameter Lie group operating on the variables (t^α, x^i). A typical element of the group is the transformation

$$\bar{t}^\alpha = \bar{t}^\alpha(t^\gamma, x^j, \alpha_s), \qquad \bar{x}^i = \bar{x}^i(t^\gamma, x^j, \alpha_s), \tag{8}$$

where $\alpha_s (s,t = 1, \ldots, r)$ represent the r parameters of the group. We assume that the identity transformation of the group is given for $\alpha_s = 0$. The infinitesimal transformations corresponding to the finite transformations given by Eq. (8) are

$$\delta t^\alpha = \tau^\alpha_{(s)}\alpha_s, \quad \delta x^i = \xi^i_{(s)}\alpha_s \qquad \text{(summation over s)}, \tag{9}$$

where

$$\tau^\alpha_{(s)} = \left[\frac{\partial \bar{t}^\alpha(t^\gamma, x^j, \alpha_t)}{\partial \alpha_s}\right]_{\alpha_t=0}, \qquad \xi^i_{(s)} = \left[\frac{\partial \bar{x}^i(t^\gamma, x^j, \alpha_t)}{\partial \alpha_s}\right]_{\alpha_t=0}.$$

Corresponding to the infinitesimal increments given by Eq. (9) we have, from Eq. (4),

$$\overset{*}{\delta} x^i = \delta x^i - \dot{x}^i_\beta \delta t^\beta = \zeta^i_{(s)} \alpha_s \quad \text{(summation over s)},$$

where

$$\zeta^i_{(s)} = \xi^i_{(s)} - \dot{x}^i_\beta \tau^\beta_{(s)} \quad \text{(summation over s)}.$$

The variations (9) induce a variation δI of the fundamental integral (2) which we evaluate according to (5). This gives

$$\delta I = \int_{G_t} \left\{ \frac{\partial L}{\partial x^i} - \frac{d}{dt^\alpha} \left(\frac{\partial L}{\partial \dot{x}^i_\alpha} \right) \right\} \eta^i_{(s)} \omega^\alpha_s d(t) + \int_{G_t} \frac{d}{dt^\beta} \left\{ L\xi^\beta_{(s)} + \frac{\partial L}{\partial \dot{x}^i_\beta} \eta^i_{(s)} \right\} \omega_s d(t). \qquad (10)$$

(summation over s)

We now require that this fundamental integral be invariant under (9) up to an "independent integral". This means that L transforms under (9) according to

$$\int_{\bar{G}_t} L(\bar{t}^\beta, \bar{x}^j, \dot{\bar{x}}^j_\beta) d(t) = \int_{G_t} \{L(t^\beta, x^j, \dot{x}^j_\beta) + \Phi\} d(t),$$

where $\Phi = \Phi(t^\beta, x^j, \dot{x}^j_\beta, \omega_s)$ is the integrand of an independent integral. We must now examine carefully the meaning of Φ because much of what follows depends on this concept.

A thorough discussion of the concept of an independent integral is given by Rund (Rund, 1966). For our purposes it is merely necessary to state that the integral of a divergence depends solely on the values of its argument functions on the boundary ∂G_t of the domain G_t over which we perform the integration. Divergences are not the only integrands which have this property. There are a large class of integrands giving rise to this type of independence. These are referred to as "independent integrals". Integrands which are divergences provide a satisfactory and relatively simple theory which is adequate for our purposes. This approach was extensively developed by Weyl (Weyl, 1935). There is another special sub-class, furnished by certain determinants, which yield a theory for multiple integral problems. This work is commonly associated with Carathéodory (Carathéodory, 1935). There is a more general theory discussed and developed by Rund (Rund, 1966, p. 250) that includes the theories of Weyl and Carathéodory as special cases. This theory

is interesting because it classifies the independent integrals by expressing their integrands as homogeneous polynomials. This is extremely useful in treating MHD stability problems for the case of broken symmetries, since, as it will be shown below, it is necessary to expand the integrands of the constraint integrals in order to treat these problems.

In order to greatly simplify the discussion of the general treatment of symmetry transformations of the type given in (9), we assume now that the functions Λ are divergences, i.e., there exist m functions $\Phi^{\beta}_{(s)}$ $(t^{\alpha},x^{j},\dot{x}^{j}_{\beta})$ such that

$$\Lambda = \Lambda_{(s)} = \frac{d\Phi^{\beta}_{(s)}}{dt^{\beta}} , \tag{11}$$

where

$$\frac{d\Phi^{\beta}_{(s)}}{dt^{\beta}} \overset{\text{def.}}{=} \frac{\partial\Phi^{\beta}_{(s)}}{\partial t^{\beta}} + \frac{\partial\Phi^{\beta}_{(s)}}{\partial x^{i}} \dot{x}^{i}_{\beta} + - - -$$

and

$$\dot{x}^{i}_{\beta} \overset{\text{def.}}{=} \frac{dx^{i}}{dt^{\beta}} .$$

If Φ depends on t^{α} only, $\frac{d\Phi}{dt^{\alpha}}$ and $\frac{\partial\Phi}{\partial t^{\alpha}}$ have the same meaning.

We also assume that Λ is linear in the parameters α_s;

$$\Lambda = \Lambda_{(s)} (t^{\beta},x^{j},\dot{x}^{j}_{\beta})\alpha_{s}. \quad \text{(summation over s)}$$

Then we obtain

$$\delta I = \int_{G_t} \Lambda_{(s)}\alpha_{s}d(t) . \tag{12}$$

Comparison with (10) yields

$$\int_{G_t}\left[\frac{d}{dt^{\beta}}\frac{\partial L}{\partial \dot{x}^{i}_{\beta}} - \frac{\partial L}{\partial x^{i}}\right]\eta^{i}_{(s)}\alpha_{s}d(t) = \int_{G_t}\left\{\frac{d}{dt^{\beta}}\left[L\xi^{\beta}_{(s)} + \frac{\partial L}{\partial \dot{x}^{i}_{\beta}}\eta^{i}_{(s)}\right] - \Lambda_{(s)}\right\}\alpha_{s}d(t)$$

(summation over s)

By hypothesis, this equation is valid for any region G_t. The α_s are the r parameters of an r-parameter Lie group and therefore must be independent. We have

$$\left[\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^i_\alpha}\right) - \frac{\partial L}{\partial x^i}\right] \eta^i_{(s)} = \frac{d}{dt^\beta}\left[L\xi^\beta_{(s)} + \frac{\partial L}{\partial \dot{x}^i_\beta}\eta^i_{(s)}\right] - \Phi_{(s)}, (s=1,\ldots,r) .$$

Combining this with Eq. (11), we have, finally,

$$\left[\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^i_\alpha}\right) - \frac{\partial L}{\partial x^i}\right] \eta^i_{(s)} = \frac{d}{dt^\beta}\left[L\xi^\beta_{(s)} + \frac{\partial L}{\partial \dot{x}^i_\beta}\eta^i_{(s)} - \Phi^\beta_{(s)}\right], (s=1,\ldots,r) . \qquad (13)$$

(summation over s)

In II we have discussed the concept of generalized gauge symmetries as developed by Calkin. We have related the corresponding constants of the motion to the linear superposable states of closed plasma structures. We will now show that Eq. (13) ties together the concepts of Noether's theorem, least constraint, and symmetry breaking (i.e., expansions of $\Phi^\beta_{(s)}$ in complete sets of functions of the spatial coordinates).

From Eq. (11) we note that there exists a system of m properly behaved functions $\Phi^\alpha = \Phi^\alpha(t^\beta, x^j)$ obeying the divergence equation. In order to simplify the discussion and make the formalism directly comparable to that developed in II, we now introduce the following change of variables

$$t^\alpha \rightarrow x^j(x_1,x_2,x_3,t),$$

where t is the time. We assume that there are N dependent functions ϕ^a of these independent variables, and these, in turn, replace the functions $x^i(t^\alpha)$ used previously

$$x^i(t^\alpha) \rightarrow \phi^a(x^j), \; (a,b, \ldots = 1, \ldots, N).$$

A detailed discussion of Noether's theorem and related problems in this notation is given by Hill (Hill, 1951). The reader is referred to this paper for a rigorous treatment of the conservation equations which we now develop.

Considering transformations that arise continuously from the identity transformation, it is sufficient to consider the infinitesimal transformations

$$x'^k = x^k + \delta x^k$$
and
$$\psi'^{\alpha}(x') = \psi^{\alpha}(x) + \delta\psi^{\alpha}(x). \tag{14}$$

The finite transformations are found by iteration. It is then easily shown (Hill, 1951, p. 258) that one can associate a differential conservation equation with each infinitesimal symmetry transformation in the form

$$\frac{d}{dx^k}\left\{\left[L\delta^k_{\ell} - \frac{\partial L}{\partial\left(\frac{\partial\psi^{\alpha}}{\partial x^k}\right)}\frac{\partial\psi^{\alpha}}{\partial x^{\ell}}\right]\delta x^{\ell} + \frac{\partial L}{\partial\left(\frac{\partial\psi^{\alpha}}{\partial x^k}\right)}\delta\psi^{\alpha} + \delta\Omega^k\right\} = 0. \tag{15}$$

Then we can write

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\underline{S} = 0, \tag{16}$$

where ρ, the "formal charge density" associated with our symmetry transformation (14) is

$$\rho = \left[L - \frac{\partial L}{\partial\dot{\psi}^{\alpha}}\frac{\partial\psi^{\alpha}}{\partial t}\right]\delta t - \frac{\partial L}{\partial\left(\frac{\partial\psi^{\alpha}}{\partial t}\right)}\delta\underline{x}\cdot\nabla\psi^{\alpha} + \frac{\partial L}{\partial\left(\frac{\partial\psi^{\alpha}}{\partial t}\right)}\delta\psi^{\alpha} + \delta\Omega_t \tag{17}$$

and the "current density" $\underline{S}$ is

$$\underline{S} = -\frac{\partial L}{\partial\nabla\psi^{\alpha}}\frac{\partial\psi^{\alpha}}{\partial t}\delta t + \left(L\delta\underline{x} - \frac{\partial L}{\partial\nabla\psi^{\alpha}}\delta\underline{x}\cdot\nabla\psi^{\alpha}\right) + \frac{\partial L}{\partial\nabla\psi^{\alpha}}\delta\psi^{\alpha} + \delta\underline{\Omega}. \tag{18}$$

Then ρ and the three components of $\underline{S}$ can be associated with the ms functions $\psi_{(s)}$ of Eq. (11). Integrating over a closed volume V with surface Σ and using Gauss's theorem, we have

$$\frac{\partial}{\partial t}\int_{\Omega} \cdot\, d(x) = -\oint_{\Sigma} \underline{S}\, d\Sigma .$$

Calkin has shown (Calkin, 1963 , Wells, 1970) that one can write a Lagrange density for an MHD fluid in the form

$$L = \frac{1}{2}\kappa_0|\vec{E}|^2 - \frac{1}{2}\mu_0^{-1}|\vec{B}|^2 + \vec{P}\cdot\vec{E} + \rho\left(\frac{\partial\chi}{\partial t} - \eta\frac{\partial\xi}{\partial t} - \frac{1}{2}|\vec{v}|^2 - \int\rho^{-2}p\,d\rho\right),$$

where $\vec{E}$ is the electric field intensity, μ_0 and κ_0 are the permeability and permittivity of the fluid, $\vec{v}$ is the velocity of the center of mass of a fluid element, ρ is the mass density, p is the scalar pressure, χ, η, ξ are the Clebsch potentials defined by

$$\vec{v} + \rho^{-1}\vec{B}\times\vec{P} = -\nabla\chi + \eta\nabla\xi$$

and $\vec{P}$ is a "polarization" vector. The vector $\vec{P}$ is in reality a type of "vector potential" that defines the current density according to

$$\vec{j} = \frac{\partial\vec{P}}{\partial t} + \nabla\times(\vec{P}\times\vec{v}) + (\nabla\cdot\vec{P})\vec{v}, \qquad = \nabla\cdot\vec{P},$$

and is the electric charge density.

Then $\vec{P}$ is defined only up to a generalized gauge transformation of the form

$$\vec{P} \to \vec{P}' + \vec{\Lambda}, \text{ where } \frac{\partial\vec{\Lambda}}{\partial t} + \nabla\times(\vec{\Lambda}\times\vec{v}) = 0 \qquad \text{and} \qquad \nabla\cdot\vec{\Lambda} = 0 .$$

The last two equations imply that $\vec{\Lambda}$ is frozen into the fluid. In II it is shown that the three vector fields that move with the fluid are $\vec{B}$, $\vec{z}$ and $\vec{v}$ where

$$\vec{z} \overset{\text{def.}}{=} \nabla\times(\vec{v} + \rho^{-1}\vec{B}\times\vec{P}).$$

Then one can make the following infinitesimal transformations:

(I) $\delta\vec{A} = 0$ The resulting finite transformations are just the canonical transformations of the fields.

(II) $\delta\vec{A} = \delta\alpha\vec{B}$

$$\vec{P} \to \vec{P}' = \vec{P} + \delta\alpha\vec{B}$$

This transformation leads to conservation of the integral

$$\int \vec{A}\cdot\vec{B}\,d\tau .$$

(III) $\delta\vec{A} = \delta\beta\vec{z}$

$$\vec{P} \to \vec{P}' = \vec{P} + \delta\beta\vec{z}$$

This leads to conservation of the integral $\int \vec{B}\cdot\vec{v}\,d\tau$.

(IV) $\delta\vec{A} = \delta\gamma\vec{v}$

This leads to conservation of the integral $\int \vec{A}\cdot\vec{v}\,d\tau$.

For given symmetry transformations arising continuously from the identity we can write the conservation law (16). The corresponding density σ and current $\underline{S}$ are found from (17) and (18) respectively. The appropriate conserved integrals (constants of the motion) are then found by the application of vector identities. In this way the conserved integrals in (II), (III), and (IV) above have been found (see II).

We now consider one of these transformations in more detail in order to relate our fundamental Eq. (13) to the principle of least constraint and the concept of symmetry breaking.

Following Calkin (Calkin, 1961, p. 88) we consider again the transformation

$$\vec{P} \to \vec{P}' = \vec{P} + \delta\alpha\vec{B} . \tag{19}$$

The lagrangian L transforms into L', where

$$L'(\psi', \frac{\partial\psi'}{\partial x'}, x') = L(\psi', \frac{\partial\psi'}{\partial x'}, x') - \delta\alpha\vec{E}\cdot\vec{B} .$$

To show that Eq. (19) leaves the action principle invariant, we write (see Calkin, 1961, p. 88)

$$- \delta\alpha\vec{E}\cdot\vec{B} = \delta\alpha(\nabla\phi\cdot\nabla\times\vec{A} + \frac{\partial\vec{A}}{\partial t}\cdot\nabla\times\vec{A})$$

in the form

$$\frac{\partial\Omega_t}{\partial t} + \nabla\cdot\underline{\Omega}.$$

After considerable algebra, we find

$$L'(\psi', \frac{\partial\psi'}{\partial x'}, x') = L(\psi', \frac{\partial\psi'}{\partial x'}, x') + \frac{\delta\alpha}{2}[\frac{\partial}{\partial t}(\vec{A}\cdot\vec{B}) + \nabla\cdot(\vec{E}\times\vec{A} + \phi\vec{B})]$$

Since the Lagrangian does not contain derivatives of the transformed variable $\vec{P}$, we have from Eq. (15),

$$\frac{\partial}{\partial t}\{\frac{\delta\alpha}{2}(\vec{A}\cdot\vec{B})\} = -\frac{\delta\alpha}{2}\{\nabla\cdot[(\vec{E}\times\vec{A}) + \phi\vec{B}]\} . \tag{20}$$

If n is the unit normal to a boundary surface Σ, then

$$\frac{\partial}{\partial t}\int_\tau\{\frac{\delta\alpha}{2}(\vec{A}\cdot\vec{B})\}d\tau + \int_\tau\frac{\delta\alpha}{2}[\nabla\cdot(\vec{E}\times\vec{A} + \phi\vec{B})]d\tau$$

$$= \frac{\partial}{\partial t}\int_\tau\{\frac{\delta\alpha}{2}(\vec{A}\cdot\vec{B})\}d\tau + \int_\Sigma\frac{\delta\alpha}{2}(\vec{E}\times\vec{A}\cdot\hat{n} + \phi\vec{B}\cdot\hat{n})d\Sigma , \tag{21}$$

where, for the transformation (II) above, we see that

$$\Omega_t = \frac{\delta\alpha}{2}(\vec{A}\cdot\vec{B}) \quad \text{and} \quad \underline{S} = \frac{\delta\alpha}{2}\{\vec{E}\times\vec{A} + \phi\vec{B}\}.$$

We see that σ plays the role of a "formal charge density" and $\underline{S}$ plays the role of a "formal current density." Thus one can discuss the current algebra of global stability if one can relate these formal charges and currents to the equilibrium and stability of the plasma (magnetofluid) inside the boundary surface Σ of Eq. (21). For nested surfaces in an infinitely conducting fluid, the surface integral goes to zero, since $\vec{v}$, $\vec{B}$, $\vec{j}$ and $\vec{A}$ are all assumed to lie in the surfaces (see discussion of boundary conditions in II) and $\int \vec{A}\cdot\vec{B}d\tau$ is a constant of the motion (see I, page 27). If both sides of Eq. (20) are multiplied by a time independent set of functions of the space coordinates (see I, page 32) arranged in a convergent series, then $\sigma \rightarrow \tilde{\sigma}$ where

$$\tilde{\sigma} \overset{\text{def.}}{=} \{f(x^k)\}\sigma \ .$$

Then

$$\frac{\partial}{\partial t}\int \tilde{\sigma} d\tau = \frac{\partial}{\partial t}\int \frac{\delta\alpha}{2} f(x^k)\ (\vec{A}\cdot\vec{B})d\tau \tag{22}$$

$$= \frac{\delta\alpha}{2}\int f(x^k)\ \frac{\partial}{\partial t}\ (\vec{A}\cdot\vec{B})d\tau = -\ \frac{\delta\alpha}{2}\int f(x^k)\{\nabla\cdot\ [(\vec{E}\times\vec{A}) + \phi\vec{B}]\}d\tau = -\ \frac{\delta\alpha}{2}\oint f(x^k)[(\vec{E}\times\vec{A})+\phi\vec{B}]\cdot\hat{n}d\Sigma \ .$$

We assume that $\vec{A}$, $\vec{B}$, and $\vec{v}$ lie on nested surfaces of constant ρ, p, and ϕ. Thus $\nabla\rho$, ∇p, and $\nabla\phi$ are assumed normal to those surfaces. Since

$$\vec{E} = -\ \vec{v}\times\vec{B} = -\ \nabla\phi\ -\ \frac{\partial\vec{A}}{\partial t}\ ,$$

$\frac{\partial\vec{A}}{\partial t}$ must be collinear with $\nabla\phi$ if surfaces of constant ϕ are to be coincident with surfaces of constant ρ, p, $\vec{A}$, $\vec{B}$ and $\vec{v}$ and if the gauge symmetry is exact (not broken). After equilibrium is achieved, $\frac{\partial\vec{A}}{\partial t}$ goes to zero. The requirement

that $\frac{\partial \vec{A}}{\partial t}$ be normal to the surfaces of constant $\vec{A}$ is relaxed in the case of broken symmetries since then $\vec{v}\times\vec{B}$ is no longer normal to the nested surfaces (there is leakage of magnetic field and mass through the surface of the plasma structure).

Returning now to Eq. (13) and Noether's theorem, we consider a special group of transformations defined by Eq. (9). If we make just the gauge transformations corresponding to $\delta\vec{\Lambda} = \delta\alpha\vec{B}$, we have from Eqs. (21) and (22)

$$\frac{\partial}{\partial t}\int \sigma d\tau = -\frac{\delta\alpha}{2}\int f(x^k)[(\vec{E}\times\vec{A}) + \phi\vec{B}]\cdot\hat{n}d\Sigma \ . \tag{23}$$

If $f(x^k)$ is a polynomial with constant term equal to unity, then one term of the integrand in Eq. (23) is $(\vec{A}\cdot\vec{B})$. If only this single term in the integrand is retained, we will say that the gauge symmetry corresponding to $\delta\vec{\Lambda} = \delta\alpha\vec{B}$ is exact. The right hand terms in (23) are then zero and σ is a constant of the motion. If other terms in $f(x^k)$ are included in the expansion of the integrand, we will say that the symmetry is broken. As we allow more terms in the expansion of $f(x^k)$, we put more constraints on the system. The more terms other than the constant term that are included, the more badly broken the symmetry. The right hand terms in Eq. (23) are no longer zero but determine the coefficients in the expansion in the following way.

Let $I = \int f(x^k)(\vec{A}\cdot\vec{B})d\tau$

then

$$\int \frac{\partial}{\partial t}\{f(x^k)(\vec{A}\cdot\vec{B})\}d\tau = -\oint f(x^k)\{E\times A\cdot n + \phi B\cdot\hat{n}\}d\Sigma.$$

and

$$\int \frac{\partial}{\partial t}\overbrace{[1 + \text{-----}\ A_n(t)]}^{f(x^k)}(\vec{A}\cdot\vec{B})d\tau = \eta(t), \text{ where } f(x^k) \text{ is a polynomial in } x^k$$

and $\eta(t)$ corresponds to the initial value of the surface integral.

Now

$$\int \frac{\partial F}{\partial t}\, d\tau = \frac{d}{dt}\int F d\tau - \oint F\vec{v}\cdot\hat{n} d\Sigma$$

for convective flow and any space-time function F (Wells and Norwood 1969, p. 27). Therefore

$$\int\frac{\partial}{\partial t}\,[\{1 + \text{---}A_n(\phi)\}\{\vec{A}\cdot\vec{B}\}]d\tau = \frac{d}{dt}\int [\{1 + \text{---}A_n(\phi)\}\{\vec{A}\cdot\vec{B}\}]d\tau$$

$$- \oint [\{1 + \text{---}A_n(\phi)\}\{\vec{A}\cdot\vec{B}\}]\vec{v}\cdot\hat{n}d\Sigma.$$

Therefore

$$\frac{d}{dt}\int[\{1 + \text{---}A_n(\phi)\}\{\vec{A}\cdot\vec{B}\}]d\tau = \eta(t) - \oint[\{1 + \text{---}A_n(\phi)\}\{\vec{A}\cdot\vec{B}\}]\vec{v}\cdot\hat{n}d\Sigma,$$

or

$$\frac{d}{dt}\int[(M_n(\phi))(\vec{A}\cdot\vec{B})]d\tau = \eta(t) - \oint [M_n(\phi)(\vec{A}\cdot\vec{B})]\vec{v}\cdot\hat{n}d\Sigma.$$

This gives

$$\Delta\int M_n(\phi)(\vec{A}\cdot\vec{B})d\tau = - \oint M_n(\phi)\{(\vec{E}\times\vec{A}) + \phi\vec{B} + (\vec{A}\cdot\vec{B})\vec{v}\}\cdot\hat{n}d\Sigma\Delta t \qquad (24)$$

We can solve Eq. (24) for $\Delta\int M_n(\vec{A}\cdot\vec{B})d\tau$. The right hand side depends on initial and boundary conditions.

Knowing $\Delta\int M_n(\phi)\vec{A}\cdot\vec{B}d\tau$ for a give Δt, we can calculate a corresponding $\Delta\psi(\phi)$ (Wells and Norwood 1969, p. 44) or $\{\Delta n(\phi), \Delta\beta(\phi), \Delta\omega(\phi), \Delta\Omega(\phi)\}$ and can then insert them into Eqs. A(30) through A(33) of Wells and Norwood to find $\Delta\vec{B}$, $\Delta\vec{j}$, $\Delta\vec{v}$, ΔP, and $\Delta\rho$ for a given Δt. This determines $\frac{\Delta\vec{B}}{\Delta t}$, etc., for a given set of initial and boundary conditions and hence gives the explicit growth-rates toward stability of the fields for given initial conditions, boundary conditions, and degree of symmetry-breaking. The method is independent of the conventional linearization restrictions and seems quite tractable for a numerical study.

It is convenient to define $M(\phi)$ in some applications as a polynomial with constant term unity. In other applications $M(\phi)$ must be expanded in some convenient complete set of functions of the electric scalar potential (see II, p. 648).

If one does not want to perform the iteration calculation indicated by Eq. (24) but, instead, wants to calculate the final cell configurations that will result for a given degree of symmetry breaking, then a modified constant of the motion must be found. In this case

$$\tilde{\sigma} \overset{\text{def}}{=} M_n(\phi)\sigma ,$$

where $M_n(\phi)$ is an appropriate complete set of functions of the electric potential, ϕ.

Then we obtain

$$\frac{\partial}{\partial t}\int \tilde{\sigma}d\tau = \frac{\partial}{\partial t}\int \frac{\delta\alpha}{2} M_n(\phi)(\vec{A}\cdot\vec{B})d\tau = -\frac{\delta\alpha}{2}\int M_n(\phi)\{\nabla\cdot[(\vec{E}\times\vec{A}) + \phi\vec{B}]\}d\tau.$$

This gives $$\frac{\partial}{\partial t}\int \tilde{\sigma}d\tau = -\int\nabla M_n(\phi)\cdot[(\vec{E}\times\vec{A}) + \phi\vec{B}]d\tau - \int\nabla\cdot M_n(\phi)[(\vec{E}\times\vec{A}) + \phi\vec{B}]d\tau$$

$$= -\int\nabla M_n(\phi)\cdot[(\vec{E}\times\vec{A}) + \phi\vec{B}]d\tau - \int_\Sigma\{M_n(\phi)[(\vec{E}\times\vec{A}) + \phi\vec{B}]\}\cdot\hat{n}d\Sigma .$$

One must now assume that during the time that the cell decays to its final lowest energy state, the time average fields $(\vec{E}\times\vec{A})$ and $\phi\vec{B}$ remain on the nested surfaces. Then for the boundary surface,

$$M_n(\phi) = \{M_n(\phi)\}_\Sigma = \text{a fixed constant.}$$

Thus

$$\frac{\partial}{\partial t}\int \tilde{\sigma}d\tau = -\int\nabla M_n(\phi)\cdot[(\vec{E}\times\vec{A}) + \phi\vec{B}]d\tau - \{M_n(\phi)\}_\Sigma\int_\Sigma [(\vec{E}\times\vec{A}) + \phi\vec{B}]\}\cdot\hat{n}d\Sigma \quad (25)$$

The first term on the right hand side drops out because $\nabla M_n(\phi)$ is normal to the nested surfaces. The second term on the right is zero because $(\vec{E}\times\vec{A})$ and $\phi\vec{B}$ lie on the surfaces.

Now consider a different group of transformations of the type defined by Eq. (9) which include not only the gauge transformation corresponding to, say, $\vec{\Lambda} = \delta\alpha\vec{B}$ but also a simultaneous transformation of the time. Then by Eq. (9)

$$\delta t = \zeta^t_\nu \quad \text{and} \quad \delta x^{\vec{P}} = \zeta^{\vec{P}}_{\vec{\Pi}} \quad \text{(summation over } \nu \text{ and } \vec{\Pi}) \tag{26}$$

where

$$\bar{t} = t + \nu \quad \text{and} \quad \bar{x}^{\vec{P}} = x^{\vec{P}} + \vec{\Pi} = \vec{P} + \delta\alpha\vec{B}.$$

One must examine the meaning of this transformation. Consider the change in a function F(x) under an infinitesimal transformation of the form

$$x'_i = f_i(x_1 \cdots x_n;\, a_1 \cdots a_r),\ i = 1, \cdots, n, \text{ (Hammermesh, p. 296).}$$

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial X_i} dx_i = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \sum_{\ell=1}^{r} u_{i\ell}(x)\delta a_\ell$$

where

$$a_{i\ell}(x) = \sum_{k=1}^{r} \left(\frac{\partial f_i}{\partial a_k}\right)_{a=0} \delta a_k.$$

Then

$$dF = \sum_{\ell=1}^{r} \delta a_\ell \left(\sum_{i=1}^{n} a_{i\ell}(x) \frac{\partial}{\partial x_i}\right) F = \sum_{\ell=1}^{r} \delta a_\ell X_\ell F.$$

The operators

$$X_\ell = \sum_{i=1}^{n} a_{i\ell}(x) \frac{\partial}{\partial x_i}$$

are the group generators.

In the particular case of the group described by Eq. (26), one has

$$d\vec{P} = \frac{\partial \vec{P}}{\partial t} \cdot$$

We have, therefore,

$$\frac{\partial \vec{P}}{\partial t} \delta\nu = \begin{cases} \vec{B}\delta\alpha \\ \vec{z}\delta\beta \\ \vec{v}\delta\gamma \end{cases} .$$

One can now easily anticipate from this relationship, derived from the properties of the group generators, the form of the Euler-Lagrange equations of the corresponding flow.

For a nonmagnetized medium moving with a velocity $\vec{v}$ which is small compared with the velocity of light

$$\nabla x\vec{B} = \frac{\partial \vec{P}}{\partial t} + \nabla x(\vec{P}x\vec{v}) + \vec{v}(\nabla\cdot\vec{P}).$$

Then

$$\frac{\partial \vec{P}}{\partial t} = \nabla x\vec{B} - [\nabla x(\vec{P}x\vec{v}) + \vec{v}(\nabla\cdot\vec{P})].$$

The requirement on the group generators is that $\frac{\partial \vec{P}}{\partial t}$ be parallel to $\vec{B}$, $\vec{z}$, and $\vec{v}$.

Then

$$\frac{\partial \vec{P}}{\partial t} = \alpha\vec{B} = \alpha\{\nabla x\vec{B} - [\nabla x(\vec{P}x\vec{v}) + \vec{v}(\nabla\cdot\vec{P})]\},$$

$$\frac{\partial \vec{P}}{\partial t} = \beta\vec{z} = \beta\{\nabla x\vec{v} + \frac{1}{\beta}\vec{B}x\vec{P}\} = [\nabla x\vec{B} - \nabla x(\vec{P}x\vec{v}) + \vec{v}(\nabla\cdot\vec{P})],$$

$$\frac{\partial \vec{P}}{\partial t} = \gamma\vec{v} = [\nabla x\vec{B} - \nabla x(\vec{P}x\vec{v}) + \vec{v}(\nabla\cdot\vec{P})]$$

where α, β, and γ are scalar constants.

These equations are trivially satisfied for a force-free collinear flow on closed nested surfaces. This is the flow described by the Euler-Lagrange equations resulting from a variation of the total energy of a closed plasma volume subject to the constants of the motion corresponding to these three symmetries.

The coefficients of the group generators then become

$$\xi^t_\nu = \left(\frac{\partial \bar{t}}{\partial \nu}\right)_{\nu=0} = 1, \text{ and } \xi^P_{\vec{\Pi}} = \left(\frac{\partial \vec{x}^P}{\partial \vec{\Pi}}\right)_{\vec{\Pi}=0} = 1,$$

We now write Eq. (13) in the form

$$\left(\frac{d}{dt^\alpha}\left(\frac{\partial L}{\partial \dot{x}^i_\alpha}\right) - \frac{\partial L}{\partial x^i}\right)\eta^i_{(s)} = -\frac{d}{dt^\beta}\left(\theta^\beta_{(s)} + \psi^\beta_{(s)}\right) \quad \text{(summation over (s)), (27)}$$

where

$$\theta^\beta_{(s)} = -L\xi^\beta_{(s)} - \frac{\partial L}{\partial \dot{x}^i_\beta}\eta^i_{(s)}.$$

From the definition of $\eta^i_{(s)}$, we have

$$\theta^\beta_{(s)} = -\left(L\delta^\beta_\alpha - \frac{\partial L}{\partial \dot{x}^i_\beta}\dot{x}^i_\alpha\right)\xi^\alpha_{(s)} - \frac{\partial L}{\partial \dot{x}^i_\beta}\zeta^i_{(s)} \quad \text{(summation over s).}$$

It is now convenient to define a Hamiltonian tensor (Rund, 1966, p. 240) in the form

$$H^\alpha_\beta(t^\gamma, x^j, p^\gamma_j) = -\delta^\alpha_\beta L(t^\gamma, x^j, \psi^j_\gamma(t^\epsilon, x^h, p^\epsilon_h)) + p^\alpha_i \psi^i_\beta(t^\epsilon, x^h, p^\epsilon_h).$$

This implies

$$\theta^\beta_{(s)} = H^\beta_\alpha \xi^\alpha_{(s)} - \frac{\partial L}{\partial \dot{x}^i_\beta}\zeta^i_{(s)}$$

From the transformations Eq. (26) and the definition of $\eta^{i}_{(s)}$ given on page 9 above

$$\eta^{\vec{P}}_{(\delta\alpha\vec{B})} = \zeta^{\vec{P}}_{(\delta\alpha\vec{B})} - \dot{\vec{P}}_t\, \xi^{t}_{(\nu)}$$

or

$$\eta^{\vec{P}}_{\Pi} = \zeta^{\vec{P}}_{\Pi} - \dot{\vec{P}}_t\, \xi^{t}_{\nu}$$

and

$$\eta = 1 - \quad t^{\cdot}$$

We then obtain

$$\theta^{\beta}_{(s)} = H^{\beta}_{t}\xi^{t}_{(s)} - \frac{\partial L}{\partial \dot{P}_{\beta}}\, \zeta^{\vec{P}}_{(s)} \qquad \text{(Summation over s)}$$

which reduces to

$$\theta^{\beta}_{(s)} = H^{\beta}_{t} - \frac{\partial L}{\partial \dot{P}_{\beta}} = H^{\beta}_{t}$$

since L does not contain terms in $\dot{P}_{\beta}$. We have assumed that $\frac{\partial \vec{P}}{\partial t}\,\delta\nu = \dot{B}\delta\alpha$

The left-hand side of Eq. (27) is zero on any extremal subspace (i.e., the Euler-Lagrange equations are obeyed). Thus, it takes the form

$$-\frac{d}{dt^{\beta}}\{H^{\beta}_{t} + \phi^{\beta}_{\vec{\Pi}}\} = 0.$$

Utilizing the Lagrange density given above, the Hamiltonian density, H, is proportional to the <u>total energy per unit volume</u> (Rund, 1966, page 300) of the fluid; i.e., we have

$$H^{4}_{4} = -\mathcal{E} \text{ where } t^{4} = t.$$

We may write

$$\frac{d}{dt}\{-\epsilon + \phi\} = 0,$$

where

$$\epsilon \overset{\text{def.}}{=} \text{energy density of the plasmoid}$$

$$-\epsilon + \phi = f(t^1, t^2, t^3) = f(x,y,z)$$

$$\int\{-\epsilon + \phi\}d\tau = \text{constant}$$

$$-E + \int\phi^{\beta}_{(s)}\, d\tau = \text{constant (summation over s)}$$

$$E \overset{\text{def.}}{=} \text{the total energy of the closed plasma structure}$$

$$\int\phi_{(s)}d\tau = E + \text{constant (summation over s)}$$

$$\int\phi_{(s)}f(\phi)d\tau = E + \text{constant.}$$

From the discussion above, it follows that the smaller the number of terms included in $f(\xi)$, the polynomial expansion of the constraint integrand, the lower the total energy of the closed plasma structure under consideration.

The arguments presented above apply specifically to the generalized gauge symmetries. For the space-time symmetries, the Lagrange density is invariant. Thus the independent integrals $\psi^{\beta}_{(s)}$ are identically zero for these transformations. The constants of the motion corresponding to the latter symmetries are also independent integrals, however, and can be written as divergences in the sense of Eq. (15). If we perform the space-time transformations, Eq. (27) can be shown (Rund, 1966, p. 296 and 297) to take the form

$$\frac{dH^{\beta}}{dt^{\beta}} = 0.$$

Any other independent integral of the form

$$\psi_{(s)} = \frac{d\psi^{\beta}_{(s)}}{dt^{\beta}} = 0$$

can be added to

$$\frac{dH^{\beta}}{dt^{\beta}} = 0 \; .$$

Since the constants of the motion corresponding to the space-time symmetries are of this form, the relationship between the total energy and the independent integrals is of the same type for these symmetries and the generalized gauge symmetries discussed above. We have here an alternate proof of the principle of least constraint (see II, page 663) <u>if</u> the total energy of the structure can be directly related to the <u>free energy</u> available to drive various types of

instabilities. This relationship must be established independently for each type of plasma structure formed for the various possible combinations of constraint integrals (see I, page 45). In I and II we have described the force-free, collinear structure as a very low energy and very stable structure. This structure corresponds to exact gauge symmetry and totally broken space-time symmetry, i.e. the constants of the motions are

$$\int \vec{A}\cdot\vec{v}\,d\tau ,$$

$$\int \vec{B}\cdot\vec{v}\,d\tau ,$$

and

$$\int \vec{A}\cdot\vec{B}\,d\tau .$$

Linear momentum, angular momentum and total energy are not conserved. The resulting Euler-Lagrange equations are found to be

$$\nabla x\vec{B} = \kappa\vec{B}$$

and

$$\vec{v} = \beta\vec{B},$$

where κ and β are scalar constants.

If we write Euler's equation for a steady state fluid in the form

$$0 = \nabla(p + \frac{1}{2}\rho v^2) + \vec{j}x\vec{B} - \rho(\vec{\zeta}x\vec{v}),$$

the force-free collinear equilibrium equation becomes

$$0 = -\nabla(p + \frac{1}{2}\rho v^2). \qquad (28)$$

It can be shown (see Shercliff,1965, p. 187) that the complete thermodynamic energy equation can be written

$$\iiint \{\rho \frac{D}{Dt} (U + \frac{v^2}{2}) - \vec{E}\cdot\vec{j} - \text{div } Q - \text{div } p\vec{v} + \text{viscous terms}\} d\tau = 0,$$

where $\iiint \vec{E}\cdot\vec{j}\, d\tau$ is the rate at which electrodynamic energy is supplied to any instabilities growing in the structure, $\int \text{div} Q d\tau$ is the heat flux available for driving instabilities, $\int \text{div } p\vec{v}\, d\tau$ is the flow work available for driving instabilities and the viscous term is the energy available for resistive mode instabilities.

From Eq. (28) and the relation

$$U = \frac{3}{2} p ,$$

where U is the internal energy of the fluid, we have

$$\nabla(\frac{2}{3} U + \frac{1}{2} \rho v^2) = 0 \tag{29}$$

for the force-free collinear structure. But the rate of change of the energy available to drive all the instabilities (the free energy for our plasma model) is

$$\frac{D}{Dt} (U + \rho\frac{v^2}{2}) = \frac{\partial}{\partial t} (U + \rho\frac{v^2}{2}) + \vec{v}\cdot\nabla(U + \frac{1}{2} \rho v^2).$$

For a steady state force-free collinear structure ,

$$\frac{\partial}{\partial t} (\frac{2}{3} U + \rho \frac{v^2}{2}) = 0.$$

This implies

$$\frac{D}{Dt}(\frac{2}{3} U + \rho \frac{v^2}{2}) = \vec{v}\cdot\nabla(\frac{2}{3} U + \frac{1}{2} \rho v^2) = 0,$$

where we have used Eq. (29). We see that for this model, the lowest lying linear plasma state has <u>no free energy</u> available to drive instabilities.

Rewriting Eq. (27) in the form

$$\frac{d}{dt}\{\mathcal{E}(s) - \mathcal{E}(s)\} = 0$$

for an extremal subspace, we see that the exact form that the integrands will take depends not only on the amount of symmetry breaking present in a given physical situation, but also on the particular representation of the independent vector fields that is utilized in writing the Lagrange density and making the Lie group of transformations represented by Eqs.(9). The number of terms retained in the expansion $f(\phi)$ is determined by how badly the assumption underlying the symmetry is violated. For example, if there is some ohmic resistance present in the fluid, then the lines of magnetic induction will slip slowly through the fluid and the symmetry represented by the transformation $\delta\vec{A} = \epsilon\alpha\vec{B}$ will be weakly broken. This amount of line slip must be represented by the terms retained in $\int f(\phi)\vec{A}\cdot\vec{B}d\tau$. If there is no slip, i.e., an ideal fluid, the symmetry is exact and $f(\phi) = 1$, etc. If one uses a special representation of the fields, for example, the representation for azimuthally symmetric fields described in II, (Wells, 1970, p. 653), then the group parameters α_s in Eqs. (9) will be different and we have an entirely different representation of the same Lie group. This will, in turn, mean that the integrands in the constraints will change and there will not be a one to one correspondence in the integrands corresponding to the same degree of symmetry breaking. This is why we have written in II, page 655, Eq. (32),

$$I_{21} = \int T(\omega\cdot P)'d\tau \sim \int \vec{A}\cdot\vec{B}d\tau$$

i.e, $\int\vec{A}\cdot\vec{B}d\tau$ replaces $\int T(\omega\cdot P)'d\tau$ but the mapping is not one to one. The resulting Euler-Lagrange equations are the same, however, (compare Eq. (38) page 656 in II with Eq. (14), page 29 of I). This must be so since the two group representations have the same group algebra. It is interesting to note that the expansion utilized by Woltjer for azimuthally symmetric systems, i.e. $\sum_{n=0}^{\infty} a_n(\omega P)^n$, meets our requirement that $\nabla f(\phi) = \nabla(\sum_{n=0}^{\infty} a_n(\omega P)^n)$ is normal to the nested surfaces.

We have shown that Eq. (27) can be written in the form

$$\int (\mathcal{E} + \Phi^{b}_{(s)}) d\tau = \text{constant} \quad \text{(summation over s)} \tag{30}$$

where $\mathcal{E}$ is the total energy density and $\Phi^{b}_{(s)}$ is the integrand of an independent integral representing the sum of the constraints. These integrands have been assumed to be linear in the parameters, α_s. Indeed, for our special problem, we found that the integrands $\Phi^{b}_{(s)}$ took the special form

$$\frac{\delta\alpha}{2} (\vec{A}\cdot\vec{B}), \quad \frac{\delta\beta}{2} (\vec{B}\cdot\vec{v}) \quad \text{and} \quad \frac{\delta\gamma}{2} (\vec{A}\cdot\vec{v})$$

where $\delta\alpha$, $\delta\beta$ and $\delta\gamma$ are proportional to the group parameters $\delta\alpha\vec{B}$, $\delta\beta\dot{z}$ and $\delta\gamma\dot{v}$.

If Eq. (27) is considered to represent a relationship between the total energy of a closed plasma configuration and the appropriate constraints on the flow, then one can vary the integrands and replace the coefficients $\frac{\delta\alpha}{2}$ etc., with a new set of parameters, the Lagrange multipliers. These will now be proportional to the original group parameters. The old parameters α_s, and the first Lie group of transformations transform the arbitrary vector fields x^i, or equivalently ξ^i, into a subspace which represents equilibrium flow, i.e., the Euler-Lagrange equations are satisfied and the left hand side of Eq. (27) is identically zero. If we write the equation for this second variation, Eq. (30) takes the form,

$$\delta^2 \int \mathcal{E}\, d\tau - \delta^2 \int \lambda_{(s)} \Phi^{B}_{(s)} d\tau = 0 \quad \text{(summation over s)} \tag{31}$$

For the case of the generalized gauge symmetries, this equation is in the form of Eq. (2.) of II, i.e.

$$\delta E + \sum_{i=1}^{4} \sum_{n=o}^{\infty} \lambda_{in}\, \delta I_{in} = 0 \tag{32}$$

For the case of space time symmetries, $\Phi^{\beta}_{(s)}$ is zero, i.e., the Lagrange density is invariant with respect to the space time transformations (Calkin, 1961). Thus, if constants of the motion corresponding to space time symmetries are to be incorporated into the variations, they must be added separately to Eq. (32). As we observed above, however, this does not affect any of the present arguments.

Equation (31) is a special case of "the problem of Lagrange" (Rund, 1966, P. 323). In this problem, the curves which afford extreme values to the fundamental integral are required to satisfy certain subsidiary conditions. These may be in terms of first order differential equations or, in our case, constraint integrals.

The Lagrange multipliers $\gamma_{(s)}$ are a new set of parameters corresponding to a second Lie group of transformations that take the equilibrium vector fields of the first variation and transform them into the special subset of fields describing the equilibria corresponding to the constrained fields. These may or may not be relatively stable, depending on the number of constraints employed. The variational calculation of Woltjer and Chandrasekhar (See I and II) is simply a method of incorporating the Lie group of transformations on $\dot{P}$ and the time into another second group of transformations of the equilibrium fields into constrained equilibrium fields.

It is interesting to note that the reason each integrand in the constraints must be expanded in an analytic series if one is to recover all possible equilibria is now quite clear. We have already determined all possible equilibria when we obtained the fundamental variational formulas, i.e. Eq. (6) and (7). When we seek constants of the motion by performing the Lie group of transformations given by Eq. (9), we are projecting out a subgroup of constrained

equilibria. When we perform the second group of transformations given by Eqs. (31) and (32), we make a second projection to a manifold corresponding to a further specialization of constraints. When we again demand all solutions of Eqs. (6) and (7), we must reverse the whole projection process by complete expansions of our constr. nt integrals. This reverse process is of practical interest only for fields close to minimum energy when one is studying the rate of change of the stable structure for weakly broken symmetries. We will illustrate these concepts in detail in Section IV.

We should note here that if one performs the second variation to solve our problem of Lagrange using many terms in an expansion of one or more of the constants of the motion (independent integrals) corresponding to the gauge symmetries, combined with a constant of the motion corresponding to say one exact space-time symmetry, the resulting energy may be higher than that resulting from another situation in which all symmetries of whatever type are nearly exact. In this sense, the energy is not a monotonically decreasing function of the number of classes of constraints employed.

The details of these transformations for the case of the exact gauge symmetries is most easily developed in terms of the "formal charge density" defined by Eq. (17) and the corresponding formal charge, Q.

In a field-theoretical formalism, a formal charge Q is defined as the space integral of the zeroth component of a local four-vector current,

$$Q(x_0) = \int dV_3 j_0(x). \tag{33}$$

These quantities appear in discussions of symmetries and broken symmetries in quantum field theory and are used in the "current-algebraic" approach to particle theories.

In quantum field theory, relations of the type given by Eq. (22) have rather troublesome convergence properties. There are other major difficulties in quantum theory which do not arise here as long as the Lagrangian and the fields remain unquantized.

In order to make clear the relationship of Eq. (22) to the formal charges defined in Eq. (33), we note that for our purposes, $j_0(x)$ in Eq. (32) is defined by Eq. (22). Then Q, the formal charge operator, is given by $\int[j_0]dV_3$ where j_0 σ. In the usual formalism, Q is actually time independent. In our case we have already proved that

$$\frac{\partial}{\partial t}\int \sigma d(x) = \frac{\partial}{\partial t}[Q] = 0. \tag{34}$$

Section III. The Lowest Lying Plasmoid States and the Charge Operator for Linear Fields

In Section II we derived the formal charge [Q], i.e., Eq. (34). [Q] can be used to perform a variational calculation of the flow equilibrium solutions for the fluid (see I). This variational calculation is equivalent to a Lie group of transformations that take the vector fields describing a general MHD flow field into a vector subspace (manifold) which describes linear (superposable) equilibrium flows.

If one defines a complex vector field

$$= \vec{u} + i\vec{B} \ ,$$

then the transformation is of the form;

$$' = f(\ ,Q).$$

$|\alpha\rangle'_{\ell}$ is a vector subspace describing equilibrium solutions that correspond to force-free, collinear flow. This subspace is defined by an eigenvalue equation of the form

$$\vec{\Lambda}\cdot\vec{\nabla}|\vec{B}\rangle = i\kappa|\vec{B}\rangle \tag{35}$$

where

$$|\vec{B}\rangle \overset{\text{def.}}{=} (i\pm\beta)\vec{B},$$

κ is a constant (see I), β is a scalar defined below, and

$$\Lambda_1 = - M_{23} + iM_{14}$$

$$\Lambda_2 = - M_{31} + iM_{24}$$

$$\Lambda_3 = - M_{12} + iM_{34}.$$

The $M_{\mu\nu}$ are defined in terms of the Maxwell stress tensor.

The electromagnetic force terms are introduced formally in the usual way as

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = f_{\mu\nu} J^\nu ,$$

where

$$f_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -E_1 \\ -B_3 & 0 & B_1 & -E_2 \\ B_2 & -B_1 & 0 & -E_3 \\ E_1 & E_2 & E_3 & 0 \end{pmatrix}$$

$$= if^{23}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + if^{31}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ if^{12}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + if^{41}\begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}$$

$$+ if^{42}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \\ 0 & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \end{pmatrix} + if^{43}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \\ 0 & 0 & +i & 0 \end{pmatrix}$$

$J^{\cdot}$ is the electromagnetic four current.

The equation for $f_{\mu\nu}$ can be written

$$f_{\mu\nu} = if^{23} M_{23} + if^{31} M_{31} + if^{12} M_{12}$$

$$+ if^{41} M_{41} + if^{42} M_{42} + if^{43} M_{43}$$

or

$$[f_{\mu\nu}] = \frac{1}{2} [if^{\mu\nu} M_{\mu\nu}].$$

The matrices $M_{\mu\nu}$ are the Hermitian and anti-Hermitian generators of Lorentz transformations (Kursunoglu, 1962, page 50). They are introduced here because they will be used later in a group theoretical discussion of the formal charge operators.

Maxwell's equations take the form,

$$\frac{\partial f_{\mu\nu}}{\partial x_\nu} = J_\mu$$

$$\frac{\partial f_{\mu\nu}}{\partial x_\rho} + \frac{\partial f_{\nu\rho}}{\partial x_\mu} + \frac{\partial f_{\rho\mu}}{\partial x_\nu} = 0.$$

Equation (35) is equivalent to the equations derived in I and II for describing the fields for exact gauge symmetries, i.e.,

$$\nabla \times \vec{v} = \kappa \vec{v} \qquad \nabla \times \vec{B} = \kappa \vec{B}$$

$$\text{and} \quad \vec{u} = \pm\beta\vec{B}$$

$$\nabla \cdot \vec{v} = 0 \qquad \nabla \cdot \vec{B} = 0$$

where κ and β are constants.

One notes that

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Lambda_3 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\Lambda_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We may now write

$$\Lambda \cdot \nabla = \Lambda_1 \partial_1 + \Lambda_2 \partial_2 + \Lambda_3 \partial_3 + \Lambda_4 \partial_4 \, .$$

One can now differentiate between the formal charge, which, in this case, explicitly di plays the generalized gauge symmetry (see II) for a conserved current, and a "differential charge operator." The formal charge operates through a variatic~al transformation to define a Lie group of transformations of the fields related to a definite symmetry. The "charge operator" is related in this case to the <u>differential operator</u> that defines the vector subspace corresponding to the transformation.

We may write

$$\vec{[illegible]}\cdot\nabla|\eta\rangle = i\kappa|\eta\rangle$$

which implies

$$|\eta\rangle = -\frac{i}{[illegible]}\vec{[illegible]}\cdot\nabla|\eta\rangle \ . \tag{36}$$

In addition to the force-free conditions, the linear superposable fields are collinear (see II).

This means that

$$\vec{u} = [illegible]\vec{B},$$

where

$$[illegible] = \frac{\vec{u}}{\vec{B}} \quad \text{(a scalar)},$$

so that we may write the defining equation for $|\eta\rangle$ as

$$[illegible] = (i \cdot [illegible])\,\vec{B}.$$

For the special case of collinear fields,

$$[illegible] = |\vec{B}\rangle. \tag{37}$$

The charge operator must produce both of the transformations, Eqs. (36) and (37). We wi ' also require that our charge operator conform to most of the other restrictions usually put onto similar operators in quantum field theory. In most cases the requirements will not have the same physical significance that they have in field theory since our theory is classical. We will, nevertheless, find the analogy very useful in our later discussion of symmetry breaking given in Section IV.

The differential operator G corresponding to the formal charge operator Q is

$$[G] = [-i/\kappa\ (\vec{\Lambda}\cdot\nabla)].$$

Then for

$$|\vec{B}> \overset{\text{def.}}{=} (i/8) \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{pmatrix}$$

$$G\ |\vec{B}> = |\vec{B}>. \qquad (38)$$

G operates in the vector space defined by $|\vec{B}>_\alpha$ since the formal charge Q transforms the space defined by all solutions of the equilibrium equation, i.e. $|\vec{B}>_\alpha$, to the special subspace corresponding to force-free collinear flow, the $|\vec{B}>_\alpha$. We have written $|\vec{B}>_\alpha$ as a three-vector in order to retain the $M_{\mu\nu}$ (Lorentz group generators). The $|\vec{B}>_\alpha$ would more properly be written in covariant form. Since this would only make the formalism more obscure, we sacrifice the elegant covariant formalism for the sake of clarification of the physical principles involved.

Section IV Broken Symmetries and Plasmoid Interactions (An Example)

One now proceeds to apply the complex operator description given by Eq. (38) to the calculation of the fields of two interacting plasmoids.

Equation (38) describes the structure of a force-free collinear plasmoid. Two of these plasmoid will superpose linearly if $\beta_1 = \beta_2$ and $\kappa_1 = \kappa_2$ for the two structures. If $\beta_1 \neq \beta_2$ or $\kappa_1 \neq \kappa_2$ or both inequalities hold, any interaction of the two plasmoids will be nonlinear and a detailed treatment of the process would involve expansion of all the Lagrange multipliers and a machine calculation of the resulting interaction (Wells and Norwood 1969). This will be treated in detail in a later paper. We are content here with an approximate treatment which we will apply directly to a problem of immediate interest, i.e., the interaction of two plasma vortex structures interacting at the center of a magnetic mirror.

The experiment is described in detail in paper II following and in (Wells and Norwood 1969). Two plasma vortex structures are fired at each other inside a magnetic mirror by conical theta-pinch guns placed inside the mirror system as shown in Fig. 1 of paper II. It has been demonstrated elsewhere (Wells and Norwood 1969) that one structure is corotational and the other countrarotational. This means in the equation expressing collinearity

$$\vec{v} = \beta \vec{B}$$

both structures do not have the same sign of β, which implies that the interaction will be nonlinear. We assume that $\kappa_1 = \kappa_2$ since it can be shown that $\kappa \simeq \frac{\omega_p}{c}$, where ω_p is the plasma frequency and c the speed of light; and the plasma parameters are the same at each end of the machine.

Thus one sees that although the vector fields describing a force-free collinear structure corresponding to unbroken generalized gauge symmetries (described in II) are linear fields, and the free noninteracting structure is built up from those fields, supplementary conditions on interacting structures produces a nonlinear interaction.

The structures meeting at the center of the mirror are not low [illegible] structures even though they are force-free. (See discussion in Appendix II.) The interacting inertial flows and magnetic fields build up a magnetic barrier or wall (Wells 1968) which isolates the two structures for times close to zero and thus justifies an approach which might be termed a "quasi-free structure" approach. One assumes that the linear theory that describes a free structure applies for times close to zero but that the initial interaction produces a small breaking of the generalized gauge symmetries. This means that [illegible], which is shown to be a constant if the gauge symmetries are not broken (see I and II), is now a function of the spatial coordinates.

We have shown,

$$\text{[illegible]} = \frac{\text{[illegible]}}{1-\text{[illegible]}_0\text{[illegible]}} .$$

The numerator and denominator go to zero at the same rate as $\text{[illegible]} = \text{[illegible]}(\mu_0\text{[illegible]})^{-\text{[illegible]}}$.

If the gauge symmetries are broken, then one can expand the constraint integrals in a complete set of functions of the space coordinates, say $M_n(\text{[illegible]})$ and $N_n(\text{[illegible]})$ (see Wells and Norwood, 1969, p. 43). This gives

$$\text{[illegible]} = \frac{\text{[illegible]}_n M_n}{1-\text{[illegible]}_0(\text{[illegible]}_n N_n)} = f(x,y,z) .$$

We will assume that experimental observation indicates that initially the symmetry breaking is small. Then $\kappa \doteq f$, i.e., f is a slowly varying function of the spacial coordinates. In order to proceed with the calculation of the initial decay rate of the interacting structures, it is necessary to show that G in Eq. (38) is the time development operator for the "quasi-free" structures.

In the quantum-mechanical description of an energy eigenstate $|u_E(t)\rangle$, one has

$$H|u_E(t_o)\rangle = E|u_E(t_o)\rangle,$$

where

$$E = \hbar\omega.$$

We then may write

$$|u_E(t)\rangle = e^{-i\omega(t-t_o)}|u_E(t_o)\rangle$$

$$= e^{-iE(t-t_o)/\hbar}|u_E(t_o)\rangle$$

which may be put into the form

$$|u_E(t)\rangle = e^{-iH(t-t_o)/\hbar}|u_E(t_o)\rangle. \tag{39}$$

This defines

$$U(t,t_o) = e^{-iH(t-t_o)/h}$$

where $U(t,t_o)$ is the unitary time development operator for the eigenstates $|u_E(t_o)\rangle$.

In a force-free collinear structure, both the velocity and magnetic field are described by differential equations of the form (Bjorgum and Godal 1952).

$$\nabla\times\vec{c} = \kappa\vec{c} \qquad \nabla\cdot\vec{c} = 0$$

$$\nabla^2\vec{c} + \kappa^2\vec{c} = 0$$

If the medium has viscous losses, the equilibrium equation for the structure can be written

$$\frac{\partial\vec{u}}{\partial t} - \vec{u}\times\nabla\times\vec{u} = \vec{F} - \frac{1}{\rho}\nabla p - \nabla\frac{u^2}{2} + \frac{\mu}{\rho}\nabla^2\vec{u},$$

where $\vec{F} = \vec{j}\times\vec{B} = 0$ (force-free) and $\frac{\mu}{\rho} = \nu \doteq$ viscosity coefficient .

From these equations one has

$$\frac{\partial\vec{u}}{\partial t} + (\nu\kappa^2)\vec{u} = 0,$$

which may be solved by the method of separation of variables

$$\vec{u} = f(t)\vec{V}(x,y,z)$$

to give

$$\vec{u} = e^{-\nu\kappa^2 t}\,\vec{w}(x,y,z).$$

This implies that

$$|\vec{u}(t)\rangle = e^{-\nu\kappa^2 t}|\vec{u}(t_0)\rangle, \tag{40}$$

but from Eq. (38)

$$\vec{\Lambda}\cdot\vec{\nabla}|\vec{B}\rangle = i\kappa|\vec{B}\rangle$$

so that in complex operator space

$$|\vec{B}(t)\rangle = \{e^{\nu[\vec{\Lambda}\cdot\nabla]^2(t-t_0)}\}|\vec{B}(t_0)\rangle, \tag{41}$$

where Equations (37) and (40) were used. Comparison of Eqs. (39) and (41) indicates that

$$U(t,t_0) = \exp\,(\nu\kappa^2 G^2)(t-t_0).$$

The value of ν can be calculated independently and is a function of the plasma temperature and density. $U(t,t_0)$ is a hermitian time development operator that is generated by $(\nu\kappa^2 G^2)$. Thus one has proved in this approximation that G^2 is the generator of the time development operator for the decaying structures.

Infinitesimal unitary transformations may be written

$$U = 1 + i\epsilon O_G,$$

where ϵ is infinitesimal and O_G is Hermitian.

An operator Ω changes under an infinitesimal unitary transformation in the following way:

$$\Omega \Rightarrow \tilde{\Omega} = (1 + i\epsilon O_G)\Omega(1 - i\epsilon O_G) = \Omega + i\epsilon[O_G,\Omega] \text{ (to first order in } \epsilon).$$

The change in the form of an operator is

$$\delta\Omega \; \tilde{\Omega} - \Omega = i[F,\Omega],$$

where

$$F \overset{\text{def}}{=} \epsilon O_G.$$

Thus the time development of a system may be visualized as a sequence of infinitesimal unitary transformations with the generator F. F is the time development operator.

If, in particular, $O_G = G$[illegible] and G[illegible] is independent of time, then (taking $t_0=0$),

$$U = \exp(\text{[illegible]} G \text{[illegible]})$$

and

$$U^{-1} = \exp(\text{[illegible]} G \text{[illegible]})^{*}.$$

The operators [illegible] G [illegible] and U commute and therefore

$$F = U(\text{[illegible]} G \text{[illegible]})U^{-1} = \text{[illegible]} G \text{[illegible]}.$$

Then defining

$$A \text{[illegible]} \overset{\text{def}}{=} \frac{(\text{[illegible]})\text{[illegible]}}{f}$$ (the time development operator if the symmetries are broken) one can write

$$\left.\frac{d(A\text{[illegible]})}{dt}\right|_{t=0} = i[\text{[illegible]} G \text{[illegible]}, A \text{[illegible]}] = i \text{[illegible]} (\text{[illegible]}) f^{-} \text{[illegible]} .$$

We note that

$$\mathrm{Re}\, i(\text{[illegible]}) \text{[illegible]} \cong \text{[illegible]} ,$$

which implies

$$i\nu\kappa^2(\Lambda\cdot\nabla)^2 f^{-2} \doteq - i\nu\kappa^2\nabla^2 f^{-2}.$$

We define

$$-\kappa^2\nabla^2 f^{-2} \overset{\text{def}}{\equiv} - \eta^2,$$

then,

$$\left.\frac{d(A^2)}{dt}\right|_{t=0} \doteq \{-\nu\kappa^2\nabla^2 f^{-2}\} = - \nu\eta^2. \tag{42}$$

Equation (42) allows a simple example of how the operator algebra can be used to roughly approximate the effects of the interaction on the structure decay rate.

One assumes that $\left.\frac{d(A^2)}{dt}\right|_{t=0}$ remains constant for times of the order of an e-folding time, then if one defines τ as the e-folding time for the structure

$$\left.\frac{d(A^2)}{dt}\right|_{\tau} \doteq - \nu\eta^2$$

$$\left.\Delta A^2(t)\right|_{\tau} \doteq - \nu\eta^2\tau$$

$$\left.\Delta A^2(t)\right|_{\tau} \quad \text{the change in } A^2(t) \text{ in time } \tau.$$

However, the expression for $|\vec{B}(t)\rangle$ is

$$|\dot{B}(t)\rangle = \exp\{-\nu\kappa^2(t-t_0)\}|\vec{B}(t_0)\rangle$$

if the symmetries are not broken, thus

$$|\dot{B}(t)\rangle = \exp\left[-\nu\{(\Lambda\cdot\nabla)^2 + \eta^2\}(t-t_0)\right]|\vec{B}(t_0)\rangle$$

if $(t-t_0) = \tau$ and

$$|\dot{B}(t)\rangle = \exp - \nu(\kappa^2 + \eta^2)(t-t_0)|\vec{B}(t_0)\rangle \tag{43}$$

if the symmetries are broken.

It has been shown elsewhere (Norwood, 1969) that $\kappa^2 \sim \frac{1}{L^2}$, where L is of the order of the diameter of the structure.

One can now make a rough estimate of the increase in the decay rate of the structures interacting at the center of the mirror system by noting that f is a very slowly varying function of (x,y,z), thus

$$\nabla^2 f^{-2} \ll \kappa^2$$

This implies that $\vec{v} \sim e^{-\nu\kappa^2 t}$, i.e., the decay rate is not altered appreciably if the symmetry is slightly broken due to the interaction.

A more precise estimate can be easily made if the amount of symmetry breaking can be estimated more accurately. Returning to $\kappa \simeq f$, the function f(x,y,z) can be evaluated if the expansions for $\alpha_n M_n(\phi)$ and $\beta_n N_n(\phi)$ are known. If these expansions are made in the form of an analytic series, the amount of symmetry breaking can be related to the number of terms retained in the expansion. For example

$$\int \alpha_n M_n(\phi)\vec{A}\cdot\vec{B}\,d\tau \overset{\text{def}}{=} \int \{1 - \sum_{n=0}^{\infty} P_n(\phi)\}\vec{A}\cdot\vec{B}\,d\tau \qquad \text{with } \alpha_n M_n(\phi) = 1 \text{ for } n = 0 \text{ and } P_n(\phi) = 0 \text{ for } n = 0,$$

where $P_n(\phi)$ is a complete set of functions of the coordinates whose sum has been normalized to unity. Then, if all terms from n=0 to n=∞ are retained, the symmetry corresponding to $\int \vec{A}\cdot\vec{B}\,d\tau$ is completely broken. If only the n=0 and n=1 terms are retained, the symmetry is weakly broken, etc. Similar expansions for $\int \beta_n N_n(\phi)\vec{B}\cdot\vec{v}\,d\tau$ and $\int \gamma_n P_n \vec{A}\cdot\vec{v}\,d\tau$ can be made. A more complete discussion of these expansions will be given in another paper.

It has been shown elsewhere that each of the integrals can be associated with one of the independent vector fields that are convected with the flow (Wells 1970). Thus a breakdown in the convection of $\vec{B}$, $\vec{v}$, or $\vec{z}$ (the generalized vorticity vector) can be estimated from experimental observation of the corresponding interactions. This, in turn, will enable one to determine the number of terms to be retained in the expansions of I_{1n}, I_{2n}, and I_{3n}, i.e.,

$$\int \alpha_{1n} M_n(\phi)\{\vec{A}\cdot\vec{B}\}d\tau, \quad \int \beta_{1n} N_n(\phi)d\tau, \quad \text{and} \quad \int \gamma_{1n} P_n(\phi)d\tau.$$

It is suggested that the "equal time commutation algebra" represented by Eq. (42) combined with a knowledge of the group algebra of the $M_{\mu\nu}$ operators allows calculation of some detailed properties of the interactions of force-free, collinear structures.

The difficulties connected with operator convergence in the case of broken symmetries for quantized operator fields (Orzalesi, 1970) does not arise in this classical theory. Our "current algebra approach" is only an analogy employed to clarify the classical field symmetries involved in global stability calculations.

V. Conclusions

Group theoretical and symmetry formalisms have been used to derive and discuss the Chandrasekhar-Woltjer-Wentzel variational approach to finding equilibrium solutions of the conservation equations of an MHD fluid plasma model. The theory has been derived from first principles utilizing the multiple integral techniques of Caratheodory and Weyl. The principle of least constraint is shown to emerge naturally and rigorously from these formalisms. This principle combined with the fundamental variational formula for multiple independent integrals, allows calculation of the global stability of closed plasma structures. The second paper of this set presents experimental verification of the most important predictions of this theory.

Appendix

Much confusion exists in the current literature about the possible values of β for a force-free structure. These are references to the solution of the magnetohydrostatic equation

$$\nabla p = \vec{j} \times \vec{B}. \qquad \text{(A 1)}$$

There is much discussion of the stability of plasma configurations which are bounded by regions of vacuum fields for which

$$\vec{j} \times B = 0 \quad \text{or} \quad (\nabla \times \vec{B}) \times \vec{B} = 0.$$

It is then noted that one must have

$$\nabla p = 0,$$

and since this condition must exist in a boundary where pressure gradients must be finite, then $p \approx 0$ and

$$\frac{p}{\frac{B^2}{2\mu_0}} = \beta \approx 0.$$

The implication is that force-free fields are associated with low β plasmas. This is true if Eq. (A 1) is the equilibrium equation for the region of the plasma under discussion.

The plasma we are discussing in this paper has the equilibrium equation

$$0 = -\nabla(p + \frac{1}{2}\rho v^2) + \vec{j} \times \vec{B} - \rho(\vec{\zeta} \times \vec{v}) \qquad \text{where } \vec{\zeta} \overset{\text{def.}}{\equiv} \nabla \times \vec{v}.$$

If the flow is force-free and collinear, this equation takes the form

$$\nabla(p + \frac{1}{2}\rho v^2) = 0$$

or $$p + \frac{1}{2}\rho v^2 = \text{constant}.$$

If the flow is an equipartition flow (the most stable structure has $\beta = \frac{1}{(\mu_0\rho)^{1/2}}$ and $\vec{v} = \frac{1}{(\mu_0\rho)^{1/2}}\vec{B}$), then one has

$$\frac{1}{2}\rho v^2 = \frac{B^2}{2\mu_0}$$

and

$$P + \frac{B^2}{2\mu_0} = \text{constant}. \qquad \text{(A 2)}$$

The integral of $\frac{B^2}{2\mu_0}$ over the surface of any fluid element <u>within the boundary</u> of the closed plasma structure under consideration must be identically zero since the body force, $\vec{j}\times\vec{B}$, is zero throughout the volume of the structure. Thus Eq. (A 2) does not define β inside the boundaries of the plasmoid. The structure must be supported by currents somewhere in the surrounding plasma which interact with the vacuum guidefield. This means that somewhere on the boundary of the plasma, presumably close to the walls of the containing vessel, there must be a transition to vacuum magnetic field if the whole configuration is to be supported free of the walls. Thus, somewhere near the walls, β must again be a definable quantity and we must have the condition

$$p + \frac{B^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

This means that the configuration consisting of the plasmoid and its surrounding plasma is limited to values of β between one and zero. It does not mean that all force-free structures in a dynamic plasma are necessarily low β structures.

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